International Journal of Theoretical Physics, Vol. 33, No. 2, 1994

Angular Correlation of Photons in e^+e^- Collision in QED: Planar Processes

E. B. Manoukian[1,2] and A. Ungkitchanukit[1]

Received March 11, 1993

The angular correlation for two monochromatic photons produced in e^+e^- collision has been calculated to lowest order in the fine-structure constant in QED for planar processes. The angular correlation is defined as the average of the cosine of the angle between the directions of momenta of the two photons. This is done by computing, in the process, the e^+e^--spin-momentum-averaged, $\gamma\gamma$-polarization-averaged conditional probability. In particular, we learn that for energies of the photons above a certain critical value, the angular correlation is strictly positive, for planar processes, indicating that the photons tend to travel in the same direction in a statistical sense for such processes.

1. INTRODUCTION

The tendency of photons to form collimated beams has attracted much attention in the literature (Manoukian, 1992; Bialynicki-Birula and Bialynicka-Birula, 1990; Mandel, 1983; Duncan and Stehle, 1987; Mostowski and Sobolewska, 1983; Richter, 1983; Stendel and Richter, 1987; Ernst and Stehle, 1968; Dicke, 1954). This result is attributed to the Bose character of photons giving them the tendency to travel in the same direction. A measure of this latter property was recently defined (Bialynicki-Birula and Bialynicka-Birula, 1990) as the average value of the cosine of the angle between the directions of momenta of any two photons. A positive correlation then would give a clear indication of the tendency of photons to travel in the same direction. A beautiful theoretical demonstration of this is given in the study of this angular correlation of photons produced by identical and, for simplicity, noninteracting (nonrecoiling) 'pointlike' (Bialynicki- Birula and Bialynicka-Birula, 1990) and, in general, not necessarily pointlike (Manoukian, 1992)

[1]Physics Department, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand.
[2]On sabbatical leave from Royal Military College, Kingston, Ontario, K7K 5LO, Canada.

0020-7748/94/0200-0339$07.00/0

atoms in a full quantum field theory (Manoukian, 1992) setting, each of which emits one photon with the same energy (monochromatic) assumed to be due to the same atomic transition. Each source is assumed separately (i.e., in isolation) to emit photons of momenta with uniform distribution for their directions. For example, for two such sources (atoms) with a denoting the interatomic distance (the separation between the 'sites' of the two photons emitted) and q denoting the energy carried away by each of the two photons, this correlation is given by (Bialynicki-Birula and Bialynicka-Birula, 1990)

$$\langle c \rangle = \left(\frac{\cos qa}{qa} - \frac{\sin qa}{(qa)^2}\right)^2 \Bigg/ \left[1 + \left(\frac{\sin qa}{qa}\right)^2\right] \tag{1}$$

The latter is strictly positive for all q, a.

Intrigued by the above result, we wanted to consider the physically interesting situation when the 'sources' are interacting in a realistic and fully quantum field theory setting. Taking this into consideration leads, in particular, to the constraint imposed on the momenta of photons produced by the 'sources' (by momentum conservation) and complicates quite a bit the argument of the existence of a positive angular correlation between photons due to their Bose character. The simplest process that we can think of that can be calculated in quantum field theory, at least to lowest order in QED, is the celebrated process of (at least) two photons produced together with anything else in e^+e^- collision: $e^+e^- \to \gamma\gamma$ + anything. The purpose of this paper is to calculate this process to lowest order in the fine-structure constant in QED. (To this order, this reduces to the classic e^+e^- annihilation into two photons.) More precisely *given* that the two photons have been produced (that is, observed) in e^+e^- collision each carrying away an energy q with a denoting a measure of the separation distance between the sites of photon emissions, we calculate the (*conditional*) e^+e^--spin-averaged, momentum-averaged, $\gamma\gamma$-polarization-averaged probability that the two photons emerge at certain angles for *planar processes* where the emerging photons momenta and their "sites" of emission fall in a plane. From this explicit expression we determine numerically the correlation $\langle c \rangle$ defined above for the two photons. In particular, we learn that for energies q of the photons above a critical value, $\langle c \rangle$ is strictly positive for planar processes. The more general case including non-planar processes as well will be studied in a forthcoming paper which hopefully may be tested experimentally.

2. COMPUTATION OF $\langle c \rangle$

To lowest order in the fine-structure constant α, the amplitude for the process $e^+e^- \to \gamma\gamma$, up to unimportant multiplicative factors for the problem at hand, is in a standard notation given by

$$\alpha^2 \bar{v}(p_2, \sigma_2)\Big\{ e_\mu(k_1, \lambda_1)\, e^{ik_1 R_1} \gamma^\mu \frac{1}{[\gamma(p_1 - k_2) + m]} \gamma^\nu e_\nu(k_2, \lambda_2)\, e^{ik_2 R_2}$$

$$+ e_\mu(k_2, \lambda_2)\, e^{ik_2 R_1} \frac{1}{[\gamma(p_1 - k_1) + m]} \gamma^\nu e_\nu(k_1, \lambda_1)\, e^{ik_1 R_2} \Big\} u(p_1, \sigma_1) \quad (2)$$

The expression in (2) is obviously symmetric in the interchange $(k_1, \lambda_1) \leftrightarrow (k_2, \lambda_2)$. Here we consider the process with $k_1^0 = k_2^0 \equiv k^0$. We also use the notation

$$\frac{k^0}{m} \equiv q, \qquad m|\mathbf{R}_1 - \mathbf{R}_2| = a \quad (3)$$

Upon taking the absolute value squared of the expression in (2), we note, in particular, that the time-dependent phase factors disappear, and by performing the spins (σ_1, σ_2) and polarization (λ_1, λ_2) averages in a straightforward though tedious manner [cf. Sokolov *et al.* (1988) for the details], we obtain the corresponding relative probability for planar processes:

$$\alpha^2 \Bigg\{ \left(\frac{m^4 (k_1 k_2)^2}{4(p_1 k_1)^2 (p_1 k_2)^2} + \frac{m^2 (k_1 k_2)}{2(p_1 k_1)(p_1 k_2)} \right)$$
$$\times [e^{iqa\cos\theta_2} e^{-iqa\cos\theta_1} + e^{iqa\cos\theta_1} e^{-iqa\cos\theta_2}]$$
$$+ \left(1 - \frac{(k_1 k_2)^2}{2(p_1 k_1)(p_1 k_2)} \right) \Bigg\} \quad (4)$$

where $\mathbf{k}_1 \cdot \mathbf{k}_2 = (k^0)^2 \cos(\theta_2 - \theta_1)$. We consider all possible processes with different incoming momenta for the e^+ and e^- pair, and perform the momentum averages over all such processes. That is, we multiply (4) by the invariant measure

$$\frac{d^3\mathbf{p}_1}{p_1^0} \frac{d^3\mathbf{p}_2}{p_2^0} \delta(p_1 + p_2 - k_1 - k_2)$$

and integrate to obtain, in a standard manner (Sokolov *et al.*, 1988), for the probability in question for planar processes:

$$\Pi(q, a, \theta_1, \theta_2)$$
$$= N(q, a)[f_1(x) + f_2(x)(e^{iqa\cos\theta_2} e^{-iqa\cos\theta_1} + e^{iqa\cos\theta_1} e^{-iqa\cos\theta_2})] \quad (5)$$

where

$$f_1(x) = \ln[2(x^2 - x)^{1/2} + 2x - 1] - \frac{(x^2 - x)^{1/2}}{x} \quad (6)$$

$$f_2(x) = \frac{1}{2x} \ln[2(x^2 - x)^{1/2} + 2x - 1]$$
$$- \frac{1}{2x^2} \left\{ (x^2 - x)^{1/2} + \frac{1}{2} \ln[2(x^2 - x)^{1/2} + 2x - 1] \right\} \quad (7)$$

$$x = \frac{q^2}{2}[1 - \cos(\theta_2 - \theta_1)] \geq 1 \tag{8}$$

and $N(q, a)$ is a normalization factor such that

$$\int_{-1}^{1} \int_{-1}^{1} d(\cos\theta_1)\, d(\cos\theta_2)\, \Theta(x - 1)\Pi(q, a, \theta_1, \theta_2) = 1 \tag{9}$$

where $\Theta(x - 1)$ is the Heaviside step function.

The constraint $x \geq 1$ is necessary to make the functions $f_1(x)$ and $f_2(x)$ real and arises as a consequence of momentum conservation:

$$k_1 + k_2 = p_1 + p_2 \tag{10}$$

which leads upon squaring and multiplying by minus one to

$$\begin{aligned} -k_1 k_2 &= m^2 + [(\mathbf{p}_1^2 + m^2)^{1/2}(\mathbf{p}_2^2 + m^2)^{1/2} - \mathbf{p}_1 \cdot \mathbf{p}_2] \\ &\geq m^2 + (\mathbf{p}_1^2 + m^2)^{1/2}(\mathbf{p}_2^2 + m^2)^{1/2} - |\mathbf{p}_1|\,|\mathbf{p}_2| \geq 2m^2 \end{aligned} \tag{11}$$

which is just the constraint $x \geq 1$, since $x = -k_1 k_2/2m^2$. Here $\Pi(q, a, \theta_1, \theta_2)$ is the conditional probability that we are seeking. The angular correlation of the two photons is defined by

$$\langle c(q, a)\rangle = \int_{-1}^{1} \int_{-1}^{1} d(\cos\theta_1)\, d(\cos\theta_2)\, \Theta(x - 1)\cos(\theta_1 - \theta_2)\, \Pi(q, a, \theta_1, \theta_2) \tag{12}$$

and is a function of q and a.

Expression (12) is studied in the next section.

3. ANALYSIS OF THE RESULTS

The angular correlation $\langle c\rangle$ gives a measure of the tendency of the two particles to travel in the same direction. A large positive (i.e., close to plus 1) correlation means that the particles have the tendency to travel in the same direction. When the correlation becomes negative, the particles are anticorrelated, and they tend to move in opposite directions. For the two-source noninteracting case, $\langle c\rangle$ as given by equation (1) is shown in Fig. 1. It can be seen that in this case $\langle c\rangle$ is always positive and $\langle c\rangle$ exhibits scaling, i.e., it is a function of only the product qa. In contrast, for the process $e^+e^- \to \gamma\gamma$, due to the scale given by the mass m of the e^+e^-, scaling does not hold for the correlation, and $\langle c\rangle$ no longer depends only on the product qa. The q and a dependences of $\langle c\rangle$ are given in Figs. 2 and 3, respectively for planar processes.

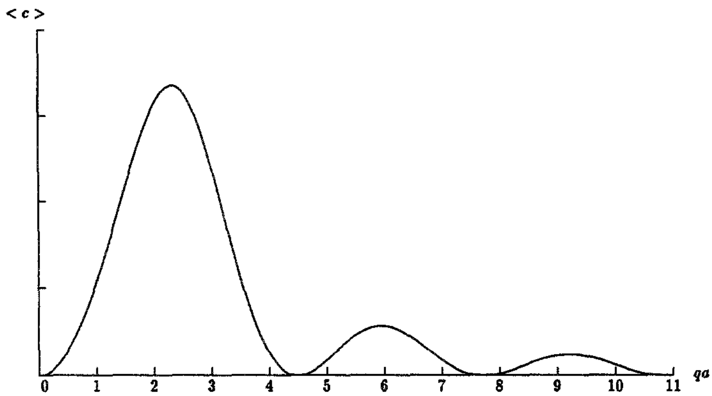

Fig. 1. Graph of $\langle c \rangle$ for the two-source noninteracting case as a function of qa.

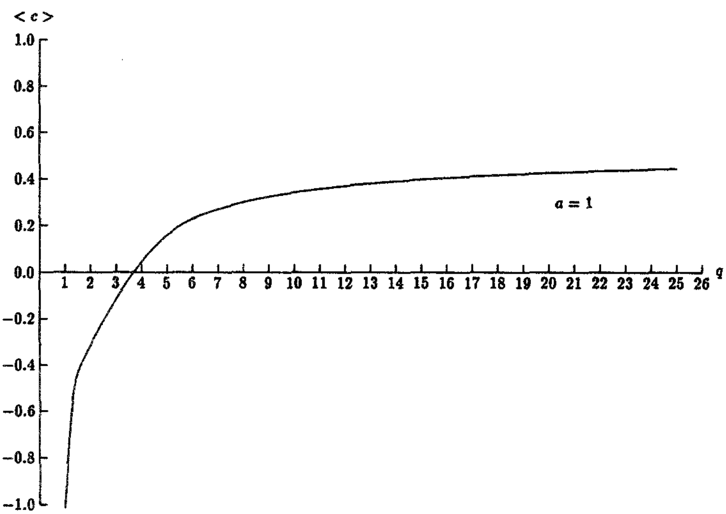

Fig. 2. Graph of $\langle c \rangle$ for the process $e^+e^- \to \gamma\gamma$ as a function of q for $a = 1$ for planar processes.

One of the main conclusions of the paper is that for sufficiently large energies $q \geq 3.7$ (corresponding to 1.9 MeV), the correlation $\langle c \rangle$ is strictly positive (independently of a) for planar processes.

The q dependence of $\langle c \rangle$ is displayed in Fig. 2 for the case $a = 1$. It can be seen that $\langle c \rangle$ increases monotonically with q, the rate of growth being very rapid for small q, where $\langle c \rangle$ is negative, and becoming much slower as q increases and $\langle c \rangle$ becomes positive, with $\langle c \rangle$ growing asymptotically to the value $+1$. This q dependence for $\langle c \rangle$ is qualitatively the

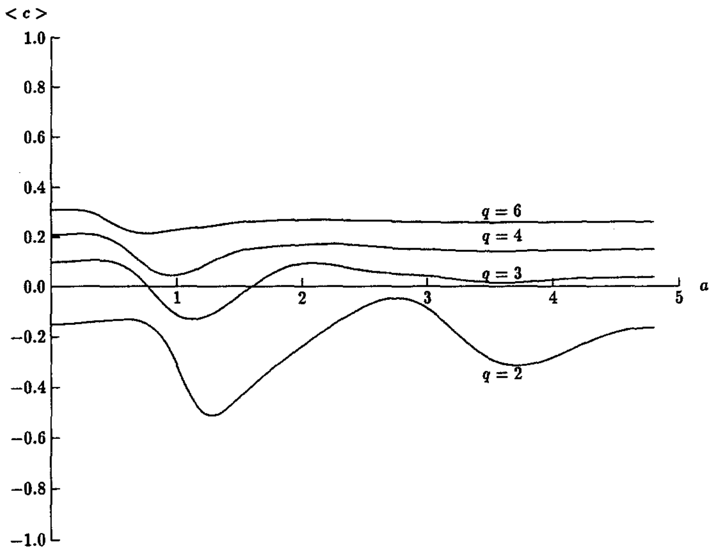

Fig. 3. Graph of $\langle c \rangle$ for the process $e^+e^- \to \gamma\gamma$ as a function of a for planar processes.

same for other values of a, but with steeper increase at negative value of $\langle c \rangle$, and for $q \approx 3.7$, $\langle c(q, a) \rangle$ becomes positive for all values of a.

The constraint given in (8) places a lower limit on the value q, $q \geq 1$, and a limit on the possible relative angle $\theta \equiv \theta_2 - \theta_1$ between the two photons. At the threshold value for q ($q = 1$), the only possible θ is $\theta = \pi$ (the photons are created back to back.) As q increases from 1, the range of possible θ, $2\Delta\theta$ (i.e., the values of possible θ spans $\pi - \Delta\theta \to \pi + \Delta\theta$), for that q increases. In fact, $\Delta\theta$ increases quite fast with q, starting at 0 when $q = 1$; $\Delta\theta$ becomes $\approx 0.26\pi$ for $q = 1.1$; $\approx 0.54\pi$ for $q = 1.5$; $\approx 0.78\pi$ for $q = 3$; and $\approx 0.94\pi$ for $q = 10$.

Although the probability function peaks around the point $\theta = \pi$, corresponding to photons emitted back to back ($\cos\theta < 0$), $\langle c \rangle$ becomes positive for q sufficiently large because the available integration domain in θ_1, θ_2 [see equation (12)] becomes dominated by the region where θ is in the range $(-\pi/2, \pi/2)$.

In Fig. 3, the a dependence of $\langle c \rangle$ is shown. The angular correlation $\langle c \rangle$ exhibits oscillations with a. The amount of oscillation decreases with increasing q. The a dependence in the conditional probability in (5) appears only in the coefficient of the function $f_2(x)$. In all cases $f_2 \ll f_1$ except when x ($x \geq 1$) is close to one, at which value they are comparable. The computation of the collision process was done within the context of QED only, and to what extent the other interactions should play a role in our conclusions (at least for small a) is not clear. Also, since our probability is a conditional probability, given that the process has occurred for a given a and the probability for such an occurrence may be very, very small (close to zero) for large a, the value of $\langle c \rangle$ may be reliable for a not too large. What about the higher-order processes contributing to $\langle c \rangle$ in QED (for example, three-photon production, e^+e^- pair production, radiative corrections, interaction of the resulting photons, etc., contributing to the probability in question in addition to two-photon production)? In such cases, the coefficient ($\equiv a_1$) of α^2 in (4) will be modified to $a_1 + O(\alpha)$, and the correlation function $\langle c \rangle$ in (12) will be modified to an expression like $c_1 + O(\alpha)$, where c_1 is given in (12). Accordingly, such higher-order processes may be safely neglected for all values of c_1 computed not too close to zero. The more general case, including non-planar processes as well, will be studied in a forthcoming paper which hopefully may be tested experimentally.

ACKNOWLEDGMENTS

This work is partially supported by the Department of National Defence of Canada. One of the authors (E.B.M.) would like to thank the Physics Department, Chulalongkorn University, for hospitality. Thanks also go to D. J. Ruffolo for help with the manuscript and figures.

REFERENCES

Bialynicki-Birula, I., and Bialynicka-Birula, Z. (1990). *Physical Review A*, **42**, 2829.
Dicke, R. H. (1954). *Physical Review*, **93**, 99.
Duncan, A., and Stehle, P. (1987). *Physical Review*, **35**, 4181.
Ernst, V., and Stehle, P. (1968). *Physical Review*, **176**, 1456.
Mandel, L. (1983). *Physical Review A*, **28**, 929.
Manoukian, E. B. (1992). *Physical Review A*, **46**, 2962.
Mostowski, J., and Sobolewska, B. (1983). *Physical Review A*, **28**, 2943.
Richter, Th. (1983). *Annalen der Physik*, **40**, 234.
Sokolov, A. A., Ternov, J. M., Zhukovski, V. Ch., and Borisov, A. V. (1988). *Quantum Electrodynamics*, Moscow.
Stendel, H., and Richter, Th. (1978). *Annalen der Physik*, **35**, 122.